3

Trapping and recycling. Statistical properties

In this chapter, we introduce two basic statistical distributions suited to an analysis of the classical inhomogeneous random walk that we introduced at the end of Chapter 2 for modelling non-ergodic cooling. These two distributions will be used throughout the book for deriving physically relevant quantities. The fact that they can be broad, with power-law tails, will also demonstrate from the beginning that Lévy statistics is naturally involved in non-ergodic cooling.

We begin in Section 3.1 by describing the evolution of the atom as a sequence of *trapping* processes of duration τ alternating with *recycling* processes of duration $\hat{\tau}$. This description will yield both physical insight and convenient calculations provided only two probability distributions are known, the distribution $P(\tau)$ of trapping times and the distribution $\hat{P}(\hat{\tau})$ of recycling times. In order to derive $P(\tau)$ and $\hat{P}(\hat{\tau})$, we then introduce in Section 3.2 simple physical models of the inhomogeneous jump rate. We then calculate $P(\tau)$ in Section 3.3 and $\hat{P}(\hat{\tau})$ in Section 3.4, using random walk techniques.

3.1 Trapping and recycling regions

As explained at the end of Chapter 2, we replace the microscopic quantum description of the evolution of the atom by a simpler description, where we consider a fictitious classical particle, completely characterized by its momentum $\mathbf{p}$, and making a random walk with a step of rms length Δp of the order of $\hbar k$. This random walk takes place in a space that can have any dimension $D = 1, 2, 3$. Subrecoil cooling is characterized by an *inhomogeneous* jump rate, depending on the position $\mathbf{p}$ in the momentum space.

The observation of the random walks of individual atoms (Fig. 2.1) suggests distinguishing two regions in momentum space, a 'trapping region' around $p = 0$ and a 'recycling region' far from $p = 0$. Indeed, when an atom reaches $p \simeq 0$ states, it can remain 'trapped' for a relatively long time. In some cases it stays

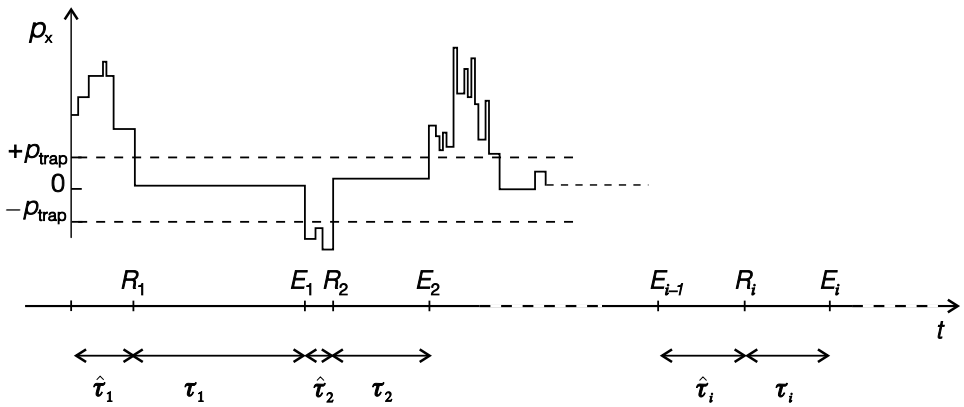

Fig. 3.1. Trapping times τ_i and recycling times $\hat{\tau}_i$. The atom returns to the trap at times $R_1, R_2, \ldots$, and exits the trap at times $E_1, E_2, \ldots$ (see Section 5.1.2).

there till the end of the laser–atom interaction. In other cases it scatters a photon before the laser is switched off which usually kicks it away from the $p \simeq 0$ region. Therefore, the atom will scatter photons again undergoing a random walk in p-space. This random walk will eventually lead the atom back to the trapping region again. Thus the atoms being kicked out of the trapping region are not lost, they are rather 'recycled' since the random walk process gives them other opportunities to reach long-lived small-p states. We introduce a momentum trap size p_{trap} to separate the two regions

$$\text{trapping region: } p \leq p_{\text{trap}}, \tag{3.1a}$$

$$\text{recycling region: } p \geq p_{\text{trap}}. \tag{3.1b}$$

The trap size p_{trap} is in principle arbitrary. We will indeed see that the physical observables no longer depend on p_{trap} in the limit $\theta \to \infty$. This trap size will be chosen conveniently below to simplify further calculations. In particular, p_{trap} will be taken to be smaller than the width p_0 of the jump rate dip (see Section 3.2):

$$p_{\text{trap}} < p_0. \tag{3.2}$$

The evolution of each atom now appears as a sequence of trapping periods of durations $\tau_1, \tau_2, \tau_3, \ldots$ alternating with recycling periods of durations $\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3, \ldots$ (see Fig. 3.1). The $\hat{\tau}_i$'s are usually called 'first return times'.

During the interaction time θ, an atom is trapped N times, N being possibly different for each atom. During θ, the same atom has therefore been recycled also N times or $N \pm 1$ times depending on whether the atom was initially (and finally)

in the trap. As we will be interested in long times θ, we have $N \gg 1$ and therefore, we consider $N \simeq N \pm 1$. If one disregards the last event[1] (either a trapping event or a recycling event) which overlaps the time $t = \theta$, the interaction time θ writes as the sum of the total trapping time T_N and the total recycling time $\hat{T}_N$:

$$\theta \simeq T_N + \hat{T}_N, \tag{3.3}$$

with

$$T_N = \sum_{i=1}^{N} \tau_i, \tag{3.4a}$$

$$\hat{T}_N = \sum_{i=1}^{N} \hat{\tau}_i. \tag{3.4b}$$

The sum T_N is the *total trapping time*, whereas $\hat{T}_N$ is the *total recycling time*, for an interaction time θ.

Both the τ_i's and the $\hat{\tau}_i$'s are independent random variables. Therefore, to study the statistical properties of the sums T_N and $\hat{T}_N$, one can think of using Central Limit Theorems (CLTs): from the probability distributions $P(\tau)$ (or $\hat{P}(\hat{\tau})$) of individual events, one infers the probability distribution $P_N(T_N)$ (or $\hat{P}_N(\hat{T}_N)$) of the sums.

A key point of the present work is that $P(\tau)$ is in many cases a 'broad' distribution, i.e. a distribution decaying so slowly at large τ that the second moment $\langle \tau^2 \rangle$ and even the first moment $\langle \tau \rangle$ are formally infinite (the same is true of $\hat{P}(\hat{\tau})$). This could be suspected from the graphical aspects of the Monte Carlo random walks of Fig. 2.1 which tend to generate very long trapping times. Usually, the finiteness of the first two moments ensures, via the CLT, that the sums T_N are distributed according to Gaussian laws ('normal' distributions) for large N. Here, the usual CLT is not directly applicable since $\langle \tau^2 \rangle$ (or even $\langle \tau \rangle$) diverges. On the other hand, if $P(\tau)$ behaves as a power law, $\tau^{-(1+\mu)}$ for large τ (which is the case here), one can use the generalized CLT of Lévy and Gnedenko. The distributions $P_N(T_N)$ no longer tend to normal distributions at large N but rather to 'Lévy distributions'.

As will be discussed below, Lévy distributions differ qualitatively from the normal distribution. For this reason, the appearance of Lévy statistics in subrecoil cooling has dramatic physical consequences. Indeed, the divergence of the average trapping time will be shown to be deeply related to the main features of the cooling mechanism, such as non-ergodicity. Thus non-ergodic cooling will appear to be qualitatively different from cooling with friction forces. In order to carry out

[1] This last event will be given a correct treatment in the quantitative calculations presented in subsequent chapters.

precise calculations, one needs to derive first the distributions $P(\tau)$ and $\hat{P}(\hat{\tau})$ of elementary events. This requires modelling of the inhomogeneous momentum random walk.

3.2 Models of inhomogeneous random walks

The distributions $P(\tau)$ and $\hat{P}(\hat{\tau})$ are determined by the random walk in momentum space. This random walk itself depends on the inhomogeneous jump rate $R(\mathbf{p})$ and on the possible existence of friction forces. In this section, we will introduce three models of inhomogeneous random walks that share the same features in the trapping region and that differ only in the recycling region. Note that we consider the random walk to be isotropic. Therefore, the jump rate $R(\mathbf{p})$ depends only on the atomic momentum modulus $p = \|\mathbf{p}\|$.

3.2.1 Friction

Before discussing both regions, we need a simple description of the friction forces that might be present. In usual laser cooling, the cooling effect of friction forces combined with the heating effect of spontaneous emission generates an approximately Gaussian stationary momentum distribution of half-width $p_{\max}$. In optimized low-intensity Doppler cooling, for instance, one has $p_{\max} \simeq (M\hbar\Gamma)^{1/2}$ where Γ^{-1} is the lifetime of the excited state. The friction forces vanish for $p \to 0$. Moreover the Gaussian momentum distribution decays very rapidly for $p > p_{\max}$. Therefore, it is reasonable to model friction forces very simply by a perfect 'wall' at $p_{\max}$ in momentum space. For $p < p_{\max}$, we consider that the atoms diffuse freely as if there was no friction, but no atomic momentum is allowed to be larger than $p_{\max}$. In other words, the real random walk with friction that explores in principle all the momentum space is, for our purposes, efficiently modelled by a standard (frictionless) random walk *confined* to a sphere of radius $p_{\max}$.

3.2.2 Trapping region

We can now establish the modelling of $R(p)$ in the trapping region, i.e. in the vicinity of $p = 0$. In all cases of non-ergodic cooling, $R(p)$ presents a dip of width p_0 around $p = 0$ which we assume to behave as a power law:

$$R(p) = \frac{1}{\tau_0}\left(\frac{p}{p_0}\right)^{\alpha}, \quad p < p_0. \tag{3.5}$$

The case of VSCPT corresponds to $\alpha = 2$ [AAK89] (see Appendix A). The flexibility of Raman cooling [KaC92] allows, in principle, any value of α. Up to

now, Raman cooling experiments have used configurations with $\alpha = 4$ and $\alpha = 2$ [RBB95]. Note that the friction forces which might be present are assumed to play no role for $p < p_0$ as p_0 will always be taken smaller than $p_{\max}$. Although the function $R(p)$ of eq. (3.5) obviously depends on a single parameter $\tau_0 p_0^\alpha$, we have introduced the value p_0 of the momentum at which the jump rate $R(p)$ saturates and takes the value τ_0^{-1} (Fig. 3.2). The parameter p_0 then characterizes the width of the dip of $R(p)$ around $p = 0$, while τ_0^{-1} is the jump rate at saturation.

Spurious mechanisms can cause the cancellation of $R(p)$ at $p = 0$ to be imperfect. In these cases, eq. (3.5) must be replaced by

$$R(p) = R_0 + \frac{1}{\tau_0}\left(\frac{p}{p_0}\right)^\alpha, \quad p < p_0. \tag{3.6}$$

In most of this book we will only consider that $R_0 = 0$. The cases $R_0 > 0$, which can be important for practical applications, can easily be taken into account with our approach. This is done in Section 7.4.

3.2.3 Recycling region

Consider now the possible models for the recycling region, i.e. the region $p > p_0$ out of the dip. In the first model, the jump rate is assumed to be constant for all $p > p_0$ and the atomic momentum is confined to a sphere of radius $p_{\max}$:

$$\text{confined model:} \quad R(p) = \frac{1}{\tau_0}, \quad p_0 \le p \le p_{\max}. \tag{3.7}$$

This *confined* model describes faithfully most situations of friction-assisted subrecoil cooling [MaA91, SHP93, WEO94, MDT94, LBS94, LKS95, HLO00].

In the second model, the jump rate is also assumed to be constant for all $p > p_0$ but the atomic momentum random walk is allowed to go to infinity ($p_{\max} \to \infty$):

$$\text{unconfined model:} \quad R(p) = \frac{1}{\tau_0}, \quad p_0 \le p. \tag{3.8}$$

This *unconfined* model is well suited to cases in which the atomic momentum diffusion is frictionless and when the optical Doppler effect, which shifts the atoms out of resonance with the cooling lasers at large p, can be neglected.

The third model assumes unconfined momentum diffusion and takes into account the decrease of the jump rate due to the *Doppler* effect:

$$\text{Doppler model:} \quad R(p) = \frac{1}{\tau_0}, \quad p_0 \le p \le p_{\mathrm{D}}, \tag{3.9a}$$

$$R(p) = \frac{1}{\tau_0}\left(\frac{p_{\mathrm{D}}}{p}\right)^2, \quad p_{\mathrm{D}} \le p, \tag{3.9b}$$

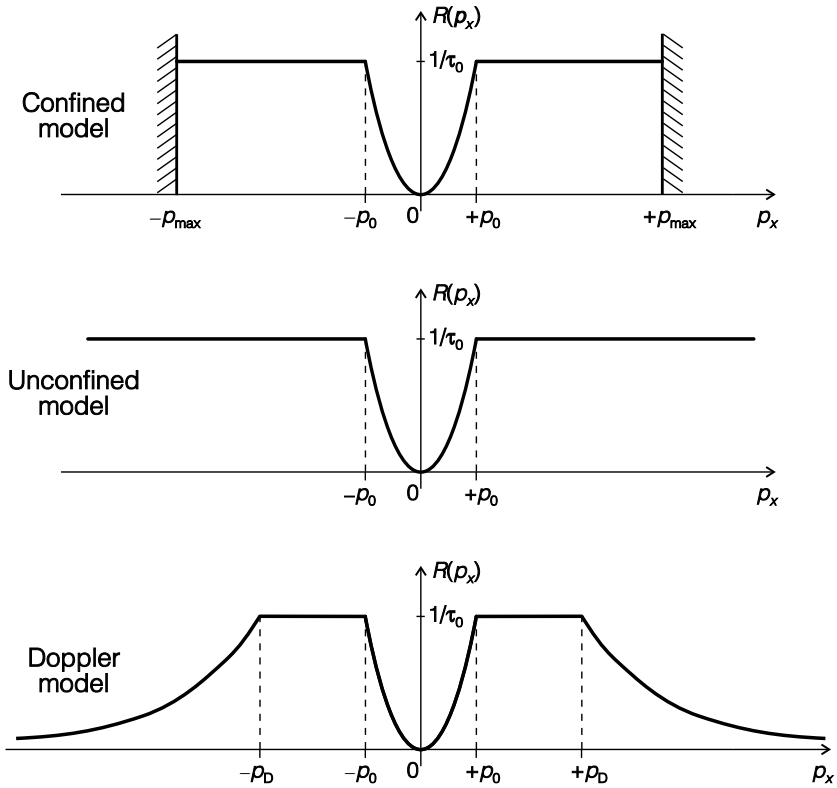

Fig. 3.2. Models of inhomogeneous random walks.

where the characteristic momentum p_{D} is defined by $kp_{\mathrm{D}}/M = \Gamma/2$ (k is the laser wave number). The previous equations for $R(p)$ are obtained by taking the small-p and the large-p limits of the Lorentzian $\Gamma^2/(\Gamma^2 + 4k^2p^2/M^2)$, describing the decrease of the jump rate due to the Doppler shift kp/M (see Appendix A, p. 152). This Doppler model describes faithfully the original one-dimensional VSCPT scheme with σ_+/σ_- polarization [AAK88].

A fourth model could be introduced, with *absorbing walls* at $p = p_{\mathrm{abs}}$. These absorbing walls would account for momentum dependent forces that appear for $p > p_{\max}$ in some experiments [LBS94] that tend to push the atoms towards *larger* momenta, unlike friction forces. This undesirable effect can in principle be reduced by an adequate choice of experimental parameters. It will therefore not be studied here.

3.2.4 Momentum jumps

The last ingredient of the random walks is the probability distribution of the momentum jumps due to spontaneous emissions. We consider that positive and negative jumps occur with the same probability (except for $p = p_{\max}$ in the confined model, cf. the above discussion of friction). The probability distribution of jumps spans an interval of approximate size $2\Delta p$ where Δp is the standard deviation of the jump lengths, the only parameter of this distribution that will be needed in this book. In most cases, Δp is of the order of the single photon momentum $\hbar k$:

$$\Delta p \simeq \hbar k. \tag{3.10}$$

One can calculate Δp precisely for specific laser cooling situations (see e.g. Section A.1.2.6, for one-dimensional σ_+/σ_- VSCPT and Section A.2.2.3, for one-dimensional Raman cooling).

3.2.5 Discussion

To sum up, physical considerations have led us to introduce three models for the inhomogeneous momentum diffusion. These models depend essentially on the parameters α, p_0 and τ_0, and possibly on R_0, $p_{\max}$ and p_{D}. They may appear to be oversimplifications of the atomic diffusion. However, as discussed below, they do grasp the essential features of subrecoil cooling. Their relevance is intimately connected to the generalized CLT: as shown in Chapter 4, the distributions of the sums T_N and $\hat{T}_N$ at large N depend *only* on the asymptotic behaviour of $P(\tau)$ and $\hat{P}(\hat{\tau})$ when these distributions are broad. So the only requirement on the models for predictions in the long time regime (large N) is that they describe correctly the *asymptotic* behaviours of $P(\tau)$ and $\hat{P}(\hat{\tau})$. Therefore, these simplified models will allow *exact* analytical predictions in the long time limit.

3.3 Probability distribution of the trapping times

3.3.1 One-dimensional quadratic jump rate

We first consider the case of a one-dimensional random walk along the p_x axis, with a quadratic variation of the jump rate around $p_x = 0$:

$$R(p_x) = \frac{1}{\tau_0}\left(\frac{p_x}{p_0}\right)^2 = \frac{1}{\tau_0}\left(\frac{p}{p_0}\right)^2. \tag{3.11}$$

Bearing in mind that the real motion takes place in a three-dimensional space, the lengths of the steps of the random walk are not all equal, but rather random between $\simeq -\Delta p$ and $\simeq +\Delta p$, corresponding to the projection of the recoil momentum (of

random direction) onto the x axis. All the points of the p_x axis can then be explored by the diffusing atom. If we now make the further assumption that the trap size p_{trap} is small compared to the step length

$$p_{\text{trap}} \ll \Delta p, \tag{3.12}$$

then all the points in the trapping region are reached with the same probability ('uniform sprinkling'). The probability density $\rho(p_x)$ that an atom entering the trap of width $2p_{\text{trap}}$ reaches the momentum p_x is therefore approximated as:

$$\rho(p_x) = \frac{1}{2\,p_{\text{trap}}}. \tag{3.13}$$

The probability for a trapped atom making a momentum jump to fall back into the trap is of the order of $p_{\text{trap}}/\Delta p$, which is negligibly small because of the inequality (3.12). The trapping time $\tau(p_x)$ for an atom landing in the trap at p_x is therefore equal to the time spent at p_x, which is directly related to the jump rate $R(p_x)$ of eq. (3.11). In other words, the jump rate $R(p_x)$ is also the *rate of escape* from the trap, for the atoms with $|p_x| < p_{\text{trap}}$.

3.3.1.1 Deterministic model

Let us first assume, for simplicity, that an atom entering the trap with a momentum p_x remains there for a well defined, deterministically fixed, time $\tau(p_x)$ given by:

$$\tau(p_x) = \frac{1}{R(p_x)} = \tau_0 \left(\frac{p_0}{p_x}\right)^2 \tag{3.14}$$

(in reality, the time τ is itself random, distributed according to an exponential law of mean $1/R(p_x)$; we shall take this into account below, see eq. (3.19)). The trapping times $\tau(p_x)$ are therefore distributed (Fig. 3.3) between a minimum value τ_{trap} (corresponding to $p_x = \pm p_{\text{trap}}$)

$$\tau_{\text{trap}} = \tau(p_{\text{trap}}) = \tau_0 \left(\frac{p_0}{p_{\text{trap}}}\right)^2 \tag{3.15}$$

and infinity (corresponding to $p_x = 0$), with a probability distribution $P(\tau)$ such that

$$P(\tau)\,|\mathrm{d}\tau| = 2\rho(p_x)\,|\mathrm{d}p_x| \tag{3.16}$$

which means that all events either between p_x and $p_x + \mathrm{d}p_x$, or between $-p_x$ and $-p_x - \mathrm{d}p_x$, contribute to trapping times between τ and $\tau + \mathrm{d}\tau$, where $|\mathrm{d}\tau|$ and $|\mathrm{d}p_x|$ are related by the equation

$$\frac{|\mathrm{d}\tau|}{|\mathrm{d}p_x|} = |\tau'(p_x)| = 2\tau_0 \frac{p_0^2}{|p_x|^3} \tag{3.17}$$

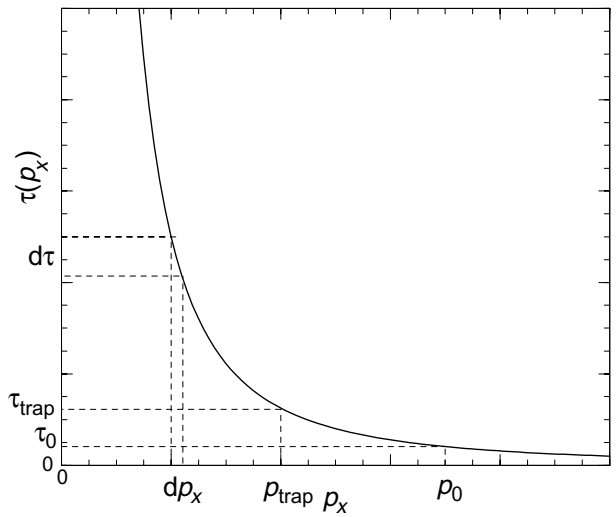

Fig. 3.3. Trapping times $\tau(p_x)$ for $p_x > 0$.

(see eq. (3.14) and Fig. 3.3). Inserting eq. (3.13) and eq. (3.17) into eq. (3.16), we then find the distribution of the trapping times to be

$$P(\tau) = \frac{\tau_{\text{trap}}^{1/2}}{2\tau^{3/2}}, \quad \tau \geq \tau_{\text{trap}}, \tag{3.18}$$

with τ_{trap} defined by eq. (3.15).

This probability distribution, shown in Fig. 3.4, is a broad function with slowly decreasing tails. The probability of observing large values of τ is so important that the average value of τ is infinite. This unusual behaviour is precisely at the root of the efficiency of the subrecoil cooling mechanisms, which are based on the existence of very long trapping times around $p_x = 0$.

3.3.1.2 Exponential model

The $\tau^{-3/2}$ behaviour of the distribution of the trapping times for large τ is the main result of the above calculation. This result is not substantially modified if one considers a more realistic model, where the trapping time for a given momentum is an exponential random variable rather than a deterministic variable. If $R(p_x)$ is the jump rate associated with a trapped momentum p_x, the conditional distribution of the trapping times for a well defined p_x is

$$P(\tau \mid p_x) = R(p_x) \exp\left(-R(p_x)\,\tau\right). \tag{3.19}$$

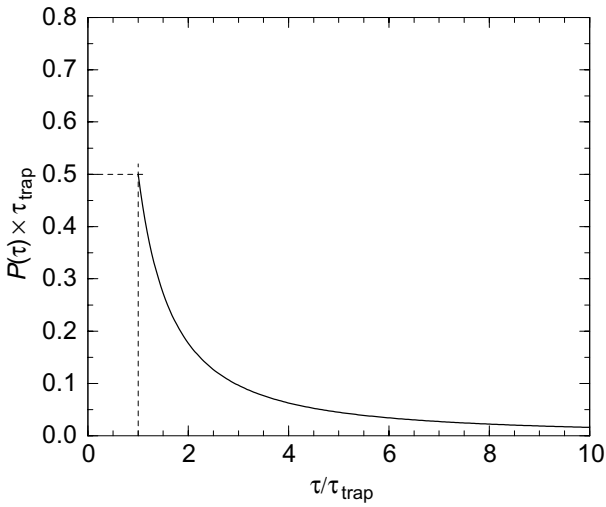

Fig. 3.4. Distribution $P(\tau)$ (deterministic model).

The total probability distribution of the trapping times is then

$$P(\tau) = \int_{-p_{\rm trap}}^{+p_{\rm trap}} P(\tau \mid p_x)\, \rho(p_x)\, \mathrm{d}p_x. \tag{3.20}$$

Using the expression for the jump rate (eq. (3.11)) and the uniform distribution for entering the trap at momentum p_x (eq. (3.13)), one finds, after changing variables to $u = p_x^2\tau/(p_0^2\tau_0)$:

$$P(\tau) = \frac{1}{2}\,\frac{\tau_{\rm trap}^{1/2}}{\tau^{3/2}}\,\gamma\left(1+\frac{1}{2}, \frac{\tau}{\tau_{\rm trap}}\right), \tag{3.21}$$

where $\gamma(\beta, x)$ is the incomplete Gamma function defined by

$$\gamma(\beta, x) = \int_0^x \mathrm{e}^{-u} u^{\beta-1}\, \mathrm{d}u. \tag{3.22}$$

Taking the limit $\tau \to \infty$, one has

$$\gamma\left(1+\frac{1}{2}, \frac{\tau}{\tau_{\rm trap}}\right) \to \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right).$$

The asymptotic behaviour of $P(\tau)$ is thus given by:

$$P(\tau) \underset{\tau \gg \tau_{\rm trap}}{\simeq} \frac{\Gamma(1/2)}{2}\,\frac{\tau_{\rm trap}^{1/2}}{2\tau^{3/2}} \tag{3.23}$$

where $\Gamma(1/2) = \sqrt{\pi}$. For finite values of τ, there are subleading correction terms which can be systematically calculated (see eq. (6.5.32) in [AbS70]).

3.3.2 Generalization to higher dimensions

If we now consider a random walk in a D-dimensional momentum space, with a quadratic jump rate still given by eq. (3.11), we can easily generalize the above results. Let us again assume that, for an atom entering the trap, the probability to land anywhere in the trapping volume $V_D(p_{\rm trap})$ is uniform. The volume $V_D(p)$ of the hypersphere of radius p reads:

$$V_D(p) = C_D p^D, \tag{3.24}$$

where C_D is the volume of the unit sphere in D dimensions:

$$C_1 = 2,\ C_2 = \pi,\ C_3 = 4\pi/3. \tag{3.25}$$

The probability $\rho(p)\,\mathrm{d}p$, for an atom landing in the trap, of landing at a momentum of modulus between p and $p + \mathrm{d}p$ is simply given by:

$$\rho(p)\mathrm{d}p = \frac{\mathrm{d}V_D(p)}{V_D(p)} = \frac{S_D p^{D-1}\,\mathrm{d}p}{V_D(p_{\rm trap})}, \tag{3.26}$$

where $S_D p^{D-1}$ is the surface of a hypersphere of radius p ($S_D = DC_D$):

$$S_1 = 2,\ S_2 = 2\pi,\ S_3 = 4\pi. \tag{3.27}$$

Thus, one obtains

$$\rho(p) = \frac{D p^{D-1}}{p_{\rm trap}^D}. \tag{3.28}$$

Calculations similar to those of the one-dimensional case then lead to

$$P(\tau) \underset{\tau \gg \tau_{\rm trap}}{\simeq} \mathcal{A}\frac{D\tau_{\rm trap}^{D/2}}{2\tau^{1+D/2}}, \tag{3.29}$$

where $\mathcal{A}$ is a numerical factor which depends on whether one assumes a deterministic (see eq. (3.14)) or an exponential (see eq. (3.19)) relation between $\tau(p)$ and $R(p)$. In the former case, $\mathcal{A} = 1$, while in the latter case, $\mathcal{A} = (D/2)\,\Gamma(D/2)$. Note that as soon as $D > 2$, the average trapping time is finite (although the variance of the trapping time still diverges if $D < 4$).

3.3.3 Generalization to a non-quadratic jump rate

If we consider more general situations where the jump rate varies as p^α (cf. eq. (3.5)), very similar calculations lead to the following result:

$$P(\tau) \underset{\tau \gg \tau_{\rm trap}}{\simeq} \mathcal{A}_\mu \frac{\mu \tau_{\rm trap}^\mu}{\tau^{1+\mu}} \qquad \text{with} \quad \mu \equiv \frac{D}{\alpha}, \tag{3.30}$$

where the characteristic trapping time τ_{trap} is defined as:

$$\tau_{\text{trap}} = \tau_0 \left(\frac{p_0}{p_{\text{trap}}} \right)^{\alpha}. \tag{3.31}$$

The numerical constant $\mathcal{A}_\mu$ is still equal to one in the deterministic case, and to $\mathcal{A}_\mu = \mu\Gamma(\mu)$ in the exponential case.

3.3.4 Discussion

We have thus shown that the asymptotic behaviour of the trapping time distribution $P(\tau)$ at large τ decays as a *power law* with an exponent μ given by the ratio of the dimension D of the momentum space to the exponent α characterizing the p-dependence of the jump rate $R(p)$ around the trapping point. This power-law distribution is conveniently written as

$$P(\tau) \underset{\tau \gg \tau_{\text{b}}}{\simeq} \frac{\mu \tau_{\text{b}}^{\mu}}{\tau^{1+\mu}} \tag{3.32}$$

with

$$\mu \equiv \frac{D}{\alpha} \qquad \text{and} \qquad \tau_{\text{b}}^{\mu} \equiv \mathcal{A}_\mu \tau_{\text{trap}}^{\mu} = \mathcal{A}_\mu \left(\frac{p_0}{p_{\text{trap}}} \right)^{D} \tau_0^{\mu}, \tag{3.33}$$

where $\mathcal{A}_\mu$ is defined by

$$\text{deterministic case:} \quad \mathcal{A}_\mu = 1, \tag{3.34a}$$

$$\text{exponential case:} \quad \mathcal{A}_\mu = \mu\Gamma(\mu). \tag{3.34b}$$

Note that, in the deterministic case, the expression (3.32) is exact for all $\tau \geq \tau_{\text{trap}}$.

When $\mu \leq 2$, the variance of τ does not exist, and the usual (Gaussian) CLT does not apply (see Chapter 4). When $\mu \leq 1$ the tails of the probability distribution are so broad that the average value of τ fails to converge. Such a situation is *a priori* favourable for efficient cooling, since it corresponds to the case where very long trapping times around $p = 0$ have a substantial probability.

In such situations, however, one cannot apply usual statistical treatments. In particular, when $\mu \leq 1$, the total trapping time T_N is clearly not proportional to $N\langle\tau\rangle$ (where N is the number of trapping events), since $\langle\tau\rangle$ is infinite. One thus has to resort to the generalized CLT of Lévy and Gnedenko (see Chapter 4). In the intermediate case $1 < \mu \leq 2$, the average trapping time is finite and differs from τ_{trap} by a factor which diverges as μ tends to one from above. For example, in the deterministic case, one gets:

$$\langle\tau\rangle = \frac{\mu}{\mu - 1}\, \tau_{\text{trap}}. \tag{3.35}$$

In order to have μ as small as possible, which is *a priori* favourable for efficient cooling, the exponent α must be large: this corresponds to 'flat' behaviour of the jump rate $R(p)$ around the trapping point. On the other hand, when the number D of dimensions increases, the tails of $P(\tau)$ decay faster, simply because of the phase space relation (eq. (3.28)), which gives less weight to small values of p when the space dimension increases.

To make the connection with real subrecoil cooling schemes (Appendix A), notice that the initial VSCPT scheme in one dimension corresponds to a broad distribution where $\langle\tau\rangle$ is infinite ($\mu = 1/2$), while in the three-dimensional case ($\mu = 3/2$) the average of τ does exist. The two-dimensional situation corresponds to the marginal case $\mu = 1$. Raman cooling corresponds to $\alpha \simeq 4$ when using Blackman pulses and to $\alpha = 2$ when using square time pulses [RBB95].

3.4 Probability distribution of the recycling times

3.4.1 Presentation of the problem: first return time in Brownian motion

In contrast to a trapping period, which consists of a single event (the atom is trapped at a given **p**), a recycling period is a random walk composed of many steps out of the trap. We characterize such a composite recycling period by a single number $\hat{\tau}$, which is the recycling time, i.e. the time needed to return to the trapping region. The aim of this section is to establish the probability distribution $\hat{P}(\hat{\tau})$ of the recycling times $\hat{\tau}$. This is in fact a 'first return time' problem, a standard problem in Brownian motion theory: $\hat{\tau}$ can be identified as the time needed for the random walk in momentum space to return to the origin.

It is well known that this problem depends crucially on the dimension of the space where Brownian motion takes place. Indeed, in the one-dimensional case, the probability that a random walker returns to the origin is equal to one (actually, the random walker returns infinitely often to its starting point). On the contrary, in dimensions greater than $D = 2$, there is a non-zero probability that the walker never returns to its starting point.

Another important parameter controlling the first return time is the average duration $1/R(p)$ of the steps of the random walk. We calculate below the recycling time distributions $\hat{P}(\hat{\tau})$ for the three models introduced in Section 3.2. In the *unconfined model* (Section 3.4.2), $R(p)$ is constant outside the trap, and the motion is the usual random walk with a uniform jump rate. In the *Doppler model* (Section 3.4.3), $R(p)$ decreases for large values of p; recycling walks reaching large values of p are slowed down, and large recycling times are obviously more probable than in the unconfined model: we therefore expect a broader distribution for the recycling times $\hat{\tau}$. On the contrary, in the *confined model* (Section 3.4.4), the random walk

motion is bounded by 'hard walls', and one expects that large recycling times are scarce, leading to a relatively narrow distribution $\hat{P}(\hat{\tau})$.

3.4.2 The unconfined model in one dimension

We first recall here a few results for the problem of an atom making a one-dimensional uniform random walk on the p_x axis (p_x is the algebraic position which takes values between $-\infty$ and $+\infty$, in contrast to its modulus $p = |p_x|$ which is always positive). The average value of the elementary step is zero (isotropic random motion) and its variance Δp^2 is independent of the position p_x. The average time between two successive steps is finite and equal to τ_0 (unconfined model). At time $t = 0$ the atom leaves the trapping region around the origin ($p \leq p_{\text{trap}}$ with $p_{\text{trap}} \ll \Delta p$). We want to determine the probability distribution $\hat{P}(\hat{\tau})$ of the time $t = \hat{\tau}$ at which the system returns *for the first time* to the trapping region.

Let us start by determining the probability distribution $P_1(n)$ of the number n of steps needed to return for the first time to the trap. At this stage, working only in terms of the *number* of steps, we deal with a purely geometric problem and the existence of long-lived trapping states for $p < p_0$ plays no role. In order to solve this problem, we introduce the probability $P_{\text{trap}}(n)$ that the atom is in the trap after n steps, *independently of the number of previous returns*. This probability is the integral over the trapping region ($-p_{\text{trap}} \leq p_x \leq p_{\text{trap}}$) of the probability density $P(p_x, n)$ of p_x after n steps. For a standard random walk, it is well known that after a large number n of steps, the distribution of p_x is Gaussian:

$$P(p_x, n) = \frac{1}{\sqrt{2\pi n\, \Delta p^2}} \exp\left(-\frac{p^2}{2\, n\, \Delta p^2}\right). \tag{3.36}$$

Using the condition $p_{\text{trap}} \ll \Delta p$, one thus finds

$$P_{\text{trap}}(n) = \int_{-p_{\text{trap}}}^{p_{\text{trap}}} \mathrm{d}p_x\, P(p_x, n) = \frac{2\, p_{\text{trap}}}{\sqrt{2\pi n}\ \Delta p}. \tag{3.37}$$

We now want to relate $P_{\text{trap}}(n)$ to the first return distribution $P_1(n)$. The atom can be in the trap after n steps, either for the first time (with probability $P_1(n)$), or because it was already in the trap after $n' < n$ steps (with probability $P_{\text{trap}}(n')$), left the trap at the step $n' + 1$ and then returned once more after $n - n'$ steps (with probability $P_1(n - n')$). All the possibilities are covered by allowing n' to vary between 1 and $n - 1$. Summing over n', we can therefore write an exact relation:

$$P_{\text{trap}}(n) = \delta_{n,0} + \sum_{n'=0}^{n} P_{\text{trap}}(n')\, P_1(n - n'), \tag{3.38}$$

where the Kronecker symbol $\delta_{n,0}$ accounts for the fact that the atom is in the trap for $n = 0$. We have extended the summation from $n' = 0$ to $n' = n$, taking into account the fact that $P_{\rm trap}(n = 0) = 1$ and $P_1(n = 0) = 0$.

One then introduces two generating functions (discrete Laplace transforms), as:

$$\mathcal{L}_{\rm d} P_{\rm trap}(s) = \sum_{n=0}^{\infty} {\rm e}^{-sn} P_{\rm trap}(n) \tag{3.39}$$

and similarly for $\mathcal{L}_{\rm d} P_1$. Multiplying eq. (3.38) by ${\rm e}^{-sn}$ and summing over n leads to:

$$\mathcal{L}_{\rm d} P_{\rm trap}(s) = 1 + \mathcal{L}_{\rm d} P_{\rm trap}(s)\, \mathcal{L}_{\rm d} P_1(s) \tag{3.40}$$

or

$$\mathcal{L}_{\rm d} P_1(s) = 1 - \frac{1}{\mathcal{L}_{\rm d} P_{\rm trap}(s)}. \tag{3.41}$$

We are interested in the long time behaviour of $\hat{P}(\hat{\tau})$. Since the average time between two steps is finite, it is obvious that the large $\hat{\tau}$ regime corresponds to a large number of steps n. The information about large $\hat{\tau}$ is thus contained in the small s behaviour of $\mathcal{L}_{\rm d} P_1(s)$. For small s, the region $n < s^{-1}$ does not contribute to leading order and the discrete sums over n can be replaced by integrals (corresponding to usual Laplace transforms). This gives, using eq. (3.36):

$$\begin{aligned} \mathcal{L}_{\rm d} P_{\rm trap}(s) &\underset{s\to 0}{\simeq} \int_0^{\infty} {\rm d}n\, {\rm e}^{-sn} P_{\rm trap}(n) \\ &= \frac{2p_{\rm trap}}{\sqrt{2\pi}\,\Delta p} \int_0^{\infty} {\rm d}n\, \frac{{\rm e}^{-sn}}{\sqrt{n}} = \frac{\sqrt{2}p_{\rm trap}}{\Delta p}\, \frac{1}{\sqrt{s}} \end{aligned} \tag{3.42}$$

(we have used $\Gamma(1/2) = \sqrt{\pi}$). It then follows from eq. (3.41) that

$$\mathcal{L}_{\rm d} P_1(s) \underset{s\to 0}{\simeq} 1 - \frac{\Delta p}{\sqrt{2}p_{\rm trap}} \sqrt{s}. \tag{3.43}$$

Note that $\mathcal{L}_{\rm d} P_1(s = 0) = \sum_{n=0}^{\infty} P_1(n) = 1$, which means that the total probability of returning to the origin is equal to one. As discussed in the next chapter, the small s behaviour of $\mathcal{L}_{\rm d} P_1(s)$ and the large n behaviour of $P_1(n)$ are linked. From eq. (4.1) and eq. (4.14) of Chapter 4, one can deduce that:

$$P_1(n) \underset{n\to\infty}{\simeq} \frac{1}{2\sqrt{2\pi}}\, \frac{\Delta p}{p_{\rm trap}}\, \frac{1}{n^{3/2}}. \tag{3.44}$$

We can now come to the time variable $\hat{\tau}$. The probability density of returning to

the trap for the first time at time $\hat{\tau}$ is related to the probability density $P_1(n)$ for the number of steps through:

$$\hat{P}(\hat{\tau}) = \sum_{n=0}^{\infty} P_1(n) P(\hat{\tau}|n), \tag{3.45}$$

where $P(\hat{\tau}|n)$ is the probability density that the n steps have taken a time $\hat{\tau}$. Since the average time between jumps is finite, the law of large numbers ensures that one can replace, to leading order in the large $\hat{\tau}$ limit, $P(\hat{\tau}|n)$ by $\delta(\hat{\tau} - n\tau_0)$ (τ_0 is the average jump time).

Using eq. (3.44), we finally obtain:

$$\hat{P}(\hat{\tau}) = \frac{\hat{\tau}_{\mathrm{b}}^{1/2}}{2\hat{\tau}^{3/2}} \qquad \text{with} \quad \hat{\tau}_{\mathrm{b}} = \frac{1}{2\pi}\left(\frac{\Delta p}{p_{\mathrm{trap}}}\right)^2 \tau_0. \tag{3.46}$$

This result, valid for large values of $\hat{\tau}$, shows that the recycling time distribution is very broad, with tails decreasing so slowly that the average value of the recycling times diverges. This calls for the use of Lévy statistics, which we shall introduce in the next chapter.

The presence of the ratio $\Delta p/p_{\mathrm{trap}}$ in expression (3.46) has an interesting interpretation. When an atom, making steps of typical size Δp, comes back in the vicinity of $p = 0$, it has a probability $p_{\mathrm{trap}}/\Delta p$ of falling into the trap. Therefore, this atom must come back typically $\Delta p/p_{\mathrm{trap}}$ times *in the vicinity of* $p = 0$ in order to have an appreciable probability to return *to the trap*. The larger Δp compared to p_{trap}, the higher the probability to 'miss' the trap when coming back to the vicinity of $p = 0$, and therefore the larger the typical return time $\hat{\tau}_{\mathrm{b}}$ to the trap.

Furthermore, the power 2 of $(\Delta p/p_{\mathrm{trap}})^2$ can also be easily understood. It comes from the fact that the m^{th} first return path ($m = \Delta p/p_{\mathrm{trap}}$) is typically m^2 times longer than the first return path (see the properties of Lévy sums, Section 4.3.1).

The return time distribution becomes even broader for higher dimensions, where the atom on a random walk has difficulty relocating its initial site. In two dimensions, $\hat{P}(\hat{\tau})$ only decays as $\hat{\tau}^{-1}\log^{-2}(\hat{\tau})$, whereas in three dimensions, there is a finite probability that the walk never returns, which corresponds to a non-zero weight of $\hat{P}(\hat{\tau})$ at $\hat{\tau} = \infty$ [Wei94].

3.4.3 The Doppler model in one dimension

We now consider the case where the jump rate $R(p)$ decreases for large values of p. We bear in mind the experiments of frictionless one-dimensional VSCPT, where the rate of fluorescence decreases as a consequence of the Doppler shift. We thus specifically take the case described by eq. (3.9), corresponding to the Lorentzian wing of the atomic fluorescence.

An exact calculation of the tail of the probability distribution $\hat{P}(\hat{\tau})$ actually turns out to be possible in this case, and is presented in Appendix B. Only a simplified argument, which reproduces the correct form of this tail, is given here.

We first notice that the probability distribution of the number of first return steps $P_1(n)$ is a purely geometrical property, independent of the duration of each step, so that the expression eq. (3.44) is still valid. The proportionality between the return *time* and the *number* of steps is, however, no longer valid. During an n steps long walk, the typical distance p_n covered by the walk is $\Delta p\sqrt{n}$, each small interval of size $\mathrm{d}p$ being visited typically $n\mathrm{d}p/(\Delta p\sqrt{n})$ times. The total time spent by the walker outside the trap can thus be approximated as:

$$\hat{\tau}(n) = \sum_{n'=1}^{n} \frac{1}{R(p_{n'})} \simeq \frac{\mathrm{d}p\sqrt{n}}{\Delta p} \sum_{i=1}^{\Delta p\sqrt{n}/\mathrm{d}p} \frac{1}{R(p_i = i\,\mathrm{d}p)} \tag{3.47}$$

since each small interval of size $\mathrm{d}p$ will contribute $\sqrt{n}\,\mathrm{d}p/\Delta p$ times. In the small $\mathrm{d}p$ limit, the sum can be replaced by an integral, and one finds using eq. (3.9):

$$\hat{\tau}(n) \simeq \frac{\sqrt{n}}{\Delta p}\,\tau_0 \int_0^{\Delta p\sqrt{n}} \frac{p^2}{p_\mathrm{D}^2}\,\mathrm{d}p \simeq \frac{\tau_0}{3}\left(\frac{\Delta p}{p_\mathrm{D}}\right)^2 n^2. \tag{3.48}$$

Note that we take the expression (3.9) for $R(p)$ even when $p_0 < p < p_\mathrm{D}$ since this region contributes negligibly to long $\hat{\tau}$'s and that the lower bound of the integral is safely extended to zero because the integral is dominated by large p's.

Finally, using the distribution $P_1(n)$ of the number of first return steps given by eq. (3.44), we obtain the distribution of the first return times as

$$\hat{P}(\hat{\tau}) = P_1(n)\frac{\mathrm{d}n}{\mathrm{d}\hat{\tau}} \simeq \frac{\hat{\tau}_\mathrm{b}^{1/4}}{4\hat{\tau}^{5/4}} \tag{3.49}$$

with now, up to prefactors of the order of one which we calculate in Appendix B,

$$\hat{\tau}_\mathrm{b} \simeq \tau_0\,\frac{\Delta p^6}{p_\mathrm{trap}^4\,p_\mathrm{D}^2}. \tag{3.50}$$

This result shows that the distribution of recycling times has very broad tails, decaying as $\hat{\tau}^{-5/4}$, i.e. still more slowly than in the case of a uniform one-dimensional random walk. This is not surprising, since when the number of steps increases (so that $\hat{\tau}$ also increases), the jump rate slows down because it explores larger values of p where the Doppler effect plays a more important role. In this case also, the average return time is infinite.

For intermediate times, the Doppler effect can be neglected and the relevant jump rate is nearly constant. Therefore the $\hat{\tau}^{-3/2}$ law describes the return time distribution for times small compared to the diffusion time associated with p_D (see Section A.1.1.5, p. 153, and Section 8.3.2).

3.4.4 The confined model: random walk with walls

We now consider the *confined model*, where the random walk in a D-dimensional space is confined by reflecting walls on a sphere $p = p_{\max}$. As for the other models, we first begin by reasoning only on the number n of steps, regardless of the time they take.

Since the motion is confined, the walk explores the sphere in a uniform way at large n. For large n, the probability of finding the atom in the trapping volume after n steps is thus simply equal to

$$P_{\text{trap}}(n) = \left(\frac{p_{\text{trap}}}{p_{\max}}\right)^D, \tag{3.51}$$

i.e. the ratio of the trapping volume to the volume of the total space. The discrete Laplace transform of this function is:

$$\mathcal{L}_{\text{d}} P_{\text{trap}}(s) = \left(\frac{p_{\text{trap}}}{p_{\max}}\right)^D \frac{1}{s}. \tag{3.52}$$

We can then obtain the probability distribution $P_1(n)$ of the number of steps for the first return times by using eq. (3.41), which is valid for all models. The Laplace transform of $P_1(n)$ is thus

$$\mathcal{L}_{\text{d}} P_1(s) = 1 - \left(\frac{p_{\max}}{p_{\text{trap}}}\right)^D s. \tag{3.53}$$

The fact that the small s expansion of $\mathcal{L}_{\text{d}} P_1(s)$ starts with a term linear in s indicates that the average number of steps needed to return to the origin is finite, and is simply equal to the coefficient of s (see eq. (4.18)):

$$\langle n \rangle = \left(\frac{p_{\max}}{p_{\text{trap}}}\right)^D. \tag{3.54}$$

We can now come to time variables. The average time τ_0 between two successive steps being finite, the average first return time $\hat{\tau}$ is now also finite (at variance with the *unconfined* and *Doppler* models), with:

$$\langle \hat{\tau} \rangle = \langle n \rangle \tau_0. \tag{3.55}$$

One thus finds:

$$\left\langle \hat{\tau} \right\rangle = \tau_0 \left(\frac{p_{\max}}{p_{\text{trap}}}\right)^D. \tag{3.56}$$

This result is important, since many experiments are carried out in a situation where diffusion out of the trap is limited by a friction mechanism. It is thus worth some further comments.

- One could actually show [Wei94] that the full distribution $\hat{P}(\hat{\tau})$ decays exponentially for large $\hat{\tau}$, as $\exp(-\hat{\tau}/\tau_{\max})$, where $\tau_{\max}$ is the time needed for the random walk to reach the wall:

$$\tau_{\max} \simeq \tau_0 \left(\frac{p_{\max}}{\Delta p} \right)^2 . \tag{3.57}$$

  Correspondingly, the result (3.56) is valid when the time is large enough so that the system can explore all the accessible space, i.e. when the evolution time is much larger than $\tau_{\max}$. In the opposite limit, the results obtained for the *unconfined model* remain valid.
- Notice that $\langle \hat{\tau} \rangle$ increases very quickly with the dimension of the space when $p_{\max} \gg p_{\text{trap}}$.
- It may appear surprising that the mean return time $\langle \hat{\tau} \rangle$ does not depend on the length Δp of the steps in the momentum space, at variance with the corresponding results for $\hat{\tau}_{\text{b}}$ in the unconfined model (cf. eq. (3.46)) and in the Doppler model (cf. eq. (3.50)). One can interpret this result by noting that when the length Δp of the individual step increases, the atom comes back faster close to the origin, but the probability of missing the trap increases because the sampling of space is coarser.
- There are several hidden assumptions in the above calculation, in particular when we have identified the average time between jumps with τ_0. This is not obvious when $p_{\text{trap}} \ll p_0$, since some jumps take place in the region where the jump rate has already substantially dropped. One can show that if $\Delta p \gg p_{\text{trap}}$, the average time between jumps remains of the order τ_0 for $D > \alpha$, while for $D < \alpha$, it is modified to:

$$\langle \tau \rangle \simeq \tau_0 \left(1 + \frac{D}{\alpha - D} \frac{p_{\text{trap}}^{D-\alpha} p_0^{\alpha}}{p_{\max}^{D}} \right). \tag{3.58}$$

3.4.5 Discussion

In this chapter, we have established some results on the statistical properties of the recycling time $\hat{\tau}$, i.e. the delay between two successive trapping periods. If the random walk out of the trap is *confined* (corresponding to a realistic situation with friction, favourable to the cooling mechanism), then the average recycling time $\langle \hat{\tau} \rangle$ is finite. For N sequences of trapping and recycling, the total time $\hat{T}_N$ spent out of the trap is then given, at large N, by the usual law of large numbers:

$$\hat{T}_N \simeq N \langle \hat{\tau} \rangle. \tag{3.59}$$

On the contrary, if the random walk is not confined, the distribution of the

recycling times is so broad that the average of $\hat{\tau}$ does not exist, and one cannot write an equation such as eq. (3.59). It will be possible, however, to determine the statistical properties of the total recycling time $\hat{T}_N$ by use of Lévy statistics, provided that one knows the asymptotic behaviour of the probability distribution for the large values of $\hat{\tau}$. In the case of a one-dimensional cooling scheme, we have obtained the asymptotic distribution of the recycling times as

$$\hat{P}(\hat{\tau}) \underset{\hat{\tau} \gg \hat{\tau}_b}{\simeq} \frac{\hat{\mu}\hat{\tau}_b^{\hat{\mu}}}{\hat{\tau}^{1+\hat{\mu}}} \tag{3.60}$$

in a form similar to eq. (3.32) for the distribution of the trapping times. We have found that

$$\hat{\mu} = \tfrac{1}{2} \tag{3.61}$$

for the case of a homogeneous random walk (*unconfined model*, with a constant delay between successive steps), and

$$\hat{\mu} = \tfrac{1}{4} \tag{3.62}$$

for the case of a jump rate decreasing as p^{-2} (*Doppler model*).

In higher dimensions, $D > 1$, the distribution of return times of an unconfined random walk to the origin becomes extremely large, and the corresponding cooling mechanism is very inefficient. The role of confining walls then becomes crucial.